



## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact [support@jstor.org](mailto:support@jstor.org).

**248. Proposed by E. T. BELL, Seattle, Washington.**

If  $u_{n+2} = 4u_{n+1} - u_n$ , with  $u_0 = 2$ ,  $u_1 = 4$ , prove that the  $\triangle_n$ , whose sides are  $u_n - 1$ ,  $u_n$ ,  $u_n + 1$ , has an integral area; also that all triangles,  $\triangle_n$ , whose areas are integers, and whose sides are consecutive integers, are given by this process. Hence show that, as  $n$  increases, the area  $\triangle_n$  approximates  $(\sqrt{3}/4)u_n^2$ , and find the degree of approximation.

SOLUTION BY WILLIAM HOOVER, Columbus, Ohio.

Integrating,

$$u_n = C_1(2 + \sqrt{3})^n + C_2(2 - \sqrt{3})^n. \quad (1)$$

When  $n = 0$ ,  $u_0 = 2$ , and when  $n = 1$ ,  $u_1 = 4$ ; then

$$2 = C_1 + C_2 \dots \quad (2)$$

$$4 = C_1(2 + \sqrt{3}) + C_2(2 - \sqrt{3}). \quad (3)$$

(2) and (3) give  $C_1 = 1$ ,  $C_2 = 1$ ; so (1) becomes

$$u_n = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n \quad (4)$$

Taking  $u_n$  as the second of three consecutive numbers,  $u_n - 1$ ,  $u_n + 1$  are the others, and the area

$$\begin{aligned} \triangle_n &= \frac{1}{4} \sqrt{3\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}^2[(2 + \sqrt{3})^n + (2 - \sqrt{3})^n]^2 - 2^2} \\ &= \frac{1}{4}\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\} \sqrt{3}\{[(7 + 4\sqrt{3})^n + (7 - 4\sqrt{3})^n] - 2\} \dots \end{aligned} \quad (5)$$

The expression under the radical sign must be shown to be a perfect square. For  $n = 1, 2$ , etc., this is the case. For the extreme case,

$$\begin{aligned} \lim_{n \rightarrow \infty} \triangle_n &= \lim_{n \rightarrow \infty} \frac{1}{4}\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}^2 \sqrt{3} \sqrt{1 - \frac{4}{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n}} \\ &= \frac{1}{4}\{(2 + \sqrt{3})^n + (2 - \sqrt{3})^n\}^2 \sqrt{3} = \frac{1}{4} \sqrt{3} u_n^2. \end{aligned}$$

**249. Proposed by CLIFFORD N. MILLS, Brookings, S. Dakota.**

A perfect number is a number which is equal to the sum of all its different divisors. In an old book on mathematics, the following method is given without proof for determining perfect numbers. The number  $2^{n-1}(2^n - 1)$  is a perfect number, if  $2^n - 1$  is a prime number. Prove the formula.

SOLUTION BY MRS. ELIZABETH BROWN DAVIS, U. S. Naval Observatory, Washington, D. C.

The sum of all the divisors of  $2^{n-1}$ , including unity, is

$$1 + 2 + 2^2 + 2^3 + \dots + 2^{n-3} + 2^{n-2} = 2^{n-1} - 1.$$

If  $2^n - 1$  is prime, the sum of all the divisors of  $2^{n-1}(2^n - 1)$  is equal to the sum of all the divisors of  $2^{n-1}$ , including unity, plus the product of the sum of these divisors into  $(2^n - 1)$ , plus  $2^{n-1}$ . Hence, the sum of all the divisors of

$$\begin{aligned} 2^{n-1}(2^n - 1) &= 2^{n-1} - 1 + (2^{n-1} - 1)(2^n - 1) + 2^{n-1} \\ &= 2 \cdot 2^{n-1} - 1 + (2^{n-1} - 1)(2^n - 1) \\ &= 2^n - 1 + (2^{n-1} - 1)(2^n - 1) \\ &= 2^{n-1}(2^n - 1). \end{aligned}$$

Therefore,  $2^{n-1}(2^n - 1)$  is a perfect number. Q. E. D.

Also solved by H. N. CARLETON, ELIJAH SWIFT, J. H. WEAVER, J. W. BALDWIN, E. B. ESCOTT, H. C. FEEMSTER, and J. W. CLAWSON.